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THE TIME TO FAILURE OF CABLES SUBJECTED TO RANDOM LOADS.(U)

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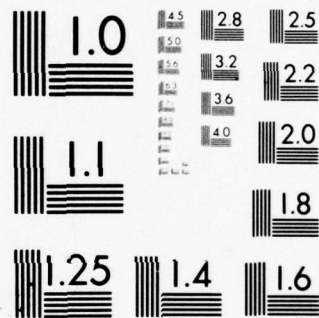
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# The Time to Failure of Cables Subjected to Random Loads

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## ABSTRACT

The effect on cable reliability of random cyclic loading such as that generated by the wave induced rocking of ocean vessels deploying these cables is examined. A simple model yielding explicit formulas is first explored. In this model, the failure time of a single element under a constant load is assumed to be exponentially distributed, and the random loadings are a two state stationary Markov process. The effect of load on failure time is assumed to follow a power law breakdown rule. In this setting, explicit results concerning the distribution of bundle or cable failure time, and especially the mean failure time, are obtained. Where the fluctuations in load are frequent relative to cable life, such as may occur in long-lived cables, it is shown that randomness in load tends to decrease mean cable life, but it is suggested that the reduction in mean life often can be restored by modestly reducing the base load on the structure or by modestly increasing the number of elements in the cable.

In later pages this simple model is extended to cover a broader range of materials and random loadings. Asymptotic distributions and mean failure times are given for cable elements that follow a Weibull distribution of failure time under constant load, and loads that are general nonnegative stationary processes subject only to some mild condition of asymptotic independence. When the power law breakdown exponent is large, the mean time to cable failure depends heavily on the exact form of the marginal probability distribution for the random load process and cannot be summarized by the first two moments of this distribution alone.

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## The Time to Failure of Cables Subjected to Random Loads

### 1. Introduction.

The Naval Undersea Center in Hawaii is involved in the construction of electromechanical cables for deep sea operation, some of which are 20,000 feet long. Kevlar-49<sup>†</sup> is being used in the strength members because of the tremendous weight savings over steel, which can hardly support its own weight in these lengths. The severing of such cables is enormously expensive because of the equipment involved on the ocean floor.

This paper examines the effect on cable reliability of random cyclic loading such as that generated by the wave induced rocking of ocean vessels deploying these cables. A simple model yielding explicit formulas is first explored. In this model, the failure time of a single element under a constant load is assumed to be exponentially distributed, and the random loadings are a two state stationary Markov process. The effect of load on failure time is assumed to follow a power law breakdown rule. In this setting, explicit results concerning the distribution of bundle or cable failure time, and especially the mean failure time, are obtained. Where the fluctuations in load are frequent relative to cable life, such as may occur in long-lived cables, it is shown that randomness in load tends to decrease mean cable life, but it is suggested that the reduction in mean life often can be restored by modestly reducing the base load on the structure or by modestly increasing the number of elements in the cable.

In later pages this simple model is extended to cover a broader range

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<sup>†</sup>Dupont trademark

of materials and random loadings. Asymptotic distributions and mean failure times are given for cable elements that follow a Weibull distribution of failure time under constant load, and loads that are general nonnegative stationary processes subject only to some mild condition of asymptotic independence. Cable behavior in these more general circumstances may differ from that found in the simple model first explored. The exact form of the marginal distribution of the stationary load process seems critical when the exponent in the power law breakdown rule is large.

## 2. A simple model with deterministic loads.

A single fiber subjected to the time varying tensile load  $\ell(t)$  fails at the random time  $T$ . We postulate the failure time distribution

$$\Pr\{T \leq t\} = 1 - \exp\{-\int_0^t K[\ell(s)]ds\},$$

which corresponds to the failure rate or hazard rate  $r(t) = K[\ell(t)]$ .

That is, a single fiber, having not failed prior to time  $t$  and carrying load  $\ell(t)$ , will fail during the interval  $[t, t+\Delta t)$  with probability  $K[\ell(t)]\Delta t + o(\Delta t)$  where  $o(\Delta t)$  denotes remainder terms of order less than  $\Delta t$  as  $\Delta t$  vanishes.

The function  $K$ , called the breakdown rule, expresses how changes in the load affect the failure probability. We concentrate on the power law breakdown rule in which  $K(\ell) = \kappa \ell^\rho$  for positive constants  $\kappa$  and  $\rho$ . Under a constant load  $\ell(t) \equiv \ell$ , the failure time of a fiber obeying power law breakdown is exponentially distributed with mean  $E[T|\ell] = 1/K(\ell) = \ell^{-\rho}/\kappa$ . A plot of mean failure time versus load is linear on log-log axes, a relationship which is commonly observed in fatigue and stress rupture studies.

Phoenix (1976) gives the following estimates for  $\rho$ :

<u>Strand type</u>	<u><math>\rho</math></u>
Kevlar-49/Epoxy	42
Graphite Fiber/Epoxy	78
S-Glass/Epoxy	30
Beryllium Wire/Epoxy	26



The analysis of cable behavior when  $\rho < 1$  involves several additional technical nuisances that do not significantly add to our understanding and that are not present when  $\rho \geq 1$ . In-as-much as Phoenix's estimates indicate that a restriction to  $\rho \geq 1$  is not severe in practice for some material types, henceforth we consider only values  $\rho \geq 1$ .

Now place  $n$  of these fibers in parallel and subject the resulting bundle or cable to a total load, constant in time, of  $nL$  units, where  $L$  is the nominal load per fiber. What is the probability distribution of the time at which the bundle fails? In the next section we will see that the affect of the constant load  $L$  and the power law breakdown coefficient  $\kappa$  can both be absorbed in a simple scale change. Therefore in the remainder of this section we concentrate on a unit load  $L = 1$  and  $\kappa = 1$ .

Since the fibers are in parallel, the bundle failure time equals the failure time of the last fiber. Let  $S_1$  be the time that the first (earliest) fiber fails, let  $S_2$  be the time that the second fiber fails, and so on. The bundle carries total load  $nL = n$ , and at the start, assuming equal load sharing, each of the  $n$  fibers carries load  $nL/n = 1$ . Any particular fiber has the survival distribution

$$\Pr\{T > t\} = e^{-K(1)t}; \quad t \geq 0,$$

and  $S_1$ , being the minimum of  $n$  such individual fiber failure times, has a survival distribution which is the  $n$ th power of this,

$$\Pr\{S_1 > t\} = e^{-nK(1)t}, \quad t \geq 0.$$

That is,  $S_1$  has an exponential distribution with parameter  $nK(1)$ .

When the first fiber fails, each of the  $n-1$  remaining fibers carries load  $nL/(n-1) = n/(n-1)$  and has the failure rate  $K[n/(n-1)]$ . Since there are  $n-1$  fibers remaining in the bundle, the rate at which the next failure in the bundle occurs is  $(n-1)$  times this, or  $(n-1)K[n/(n-1)]$ , which leads to

$$\Pr\{S_2 - S_1 > t\} = e^{-(n-1)K[n/(n-1)]t}, \quad t \geq 0.$$

This reasoning continues, and we deduce that, when  $i$  fibers have failed, the  $n-i$  remaining fibers each carry load  $n/(n-i)$  and have individual failure rates  $K[n/(n-i)]$ . The rate at which the next failure occurs among the  $n-i$  fibers remaining in the bundle is

$$\Pr\{S_{i+1} - S_i > t\} = e^{-(n-i)K[n/(n-i)]t}, \quad t \geq 0.$$

This analysis allows us to write the bundle failure time  $S_n$  as a sum of independent and exponentially distributed differences  $Y_1 = S_n - S_{n-1}$ ,  $Y_2 = S_{n-1} - S_{n-2}, \dots, Y_n = S_1 - S_0$  where  $S_0 = 0$ . That is

$$S_n = Y_1 + \dots + Y_n$$

where

$$\Pr\{Y_k > t\} = \exp\{-n^{\rho_k} k^{1-\rho_k} t\}, \quad t \geq 0.$$

We have

$$E[Y_k] = n^{-\rho} k^{\rho-1},$$

$$\text{Var}[Y_k] = n^{-2\rho} k^{2\rho-2},$$

$$\begin{aligned} E[\text{Exp}\{-sY_k\}] &= n^{\rho} k^{1-\rho} / (s + n^{\rho} k^{1-\rho}) \\ &= 1/(1 + sk^{\rho-1}/n^{\rho}) \quad \text{for } s > -n^{\rho} k^{1-\rho}, \end{aligned}$$

and

$$E[S_n] = \sum_{k=1}^n (k/n)^{\rho-1} (1/n) \quad (2.1)$$

$$\text{Var}[S_n] = (1/n) \sum_{k=1}^n (k/n)^{2\rho-2} (1/n) \quad (2.2)$$

$$E[\exp\{-sS_n\}] = 1/\prod_{k=1}^n [1 + s(k/n)^{\rho-1} (1/n)], \text{ for } s > -1. \quad (2.3)$$

Asymptotic distribution. Study of the limiting distribution of  $S_n$  as  $n$  becomes large divides into several cases according to the value of  $\rho$ . The simplest case is, at the same time, the most important case in practice and concerns  $\rho \geq 1$ . We concentrate on it. Then

$$\begin{aligned} \lim_{n \rightarrow \infty} E[S_n] &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\frac{k}{n}\right)^{\rho-1} \frac{1}{n} \\ &= \int_0^1 x^{\rho-1} dx = 1/\rho, \end{aligned}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} n \text{Var}[S_n] &= \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\frac{k}{n}\right)^{2\rho-2} \frac{1}{n} \\ &= \int_0^1 x^{2\rho-2} dx = 1/(2\rho-1), \end{aligned}$$

which suggests that  $Z_n = \sqrt{n}(S_n - 1/\rho)$  should have a limiting distribution whose mean is zero and whose variance equals  $1/(2\rho-1)$ . Indeed, this is the case, the limiting distribution being normal (Gaussian). This may be



established via Liapounov's central limit theorem or by a direct argument invoking Levy's convergence theorem for moment generating functions. We proceed with the latter approach. First

$$\begin{aligned}
 E[\exp\{-s\sqrt{n}(S_n - \sum_{k=1}^n (\frac{k}{n})^{\rho-1} \frac{1}{n})\}] &= \\
 &= \frac{\prod_{k=1}^n \exp\{s\sqrt{n} (\frac{k}{n})^{\rho-1} \frac{1}{n}\}}{\prod_{k=1}^n \{1 + s\sqrt{n} (\frac{k}{n})^{\rho-1} \frac{1}{n}\}} \\
 &= \prod_{k=1}^n \left\{ \frac{1 + s\sqrt{n} (\frac{k}{n})^{\rho-1} \frac{1}{n} + \frac{1}{2}s^2 (\frac{k}{n})^{2\rho-2} \frac{1}{n} + o(n^{-3/2})}{1 + s\sqrt{n} (\frac{k}{n})^{\rho-1} \frac{1}{n}} \right\} \\
 &= \prod_{k=1}^n \{1 + \frac{1}{2}s^2 (\frac{k}{n})^{2\rho-2} \frac{1}{n} + o(n^{-3/2})\} \\
 &\xrightarrow{n \rightarrow \infty} \exp\{\frac{1}{2}s^2 \int_0^1 x^{2\rho-2} dx\} = \exp\{\frac{1}{2}s^2/(2\rho-1)\}.
 \end{aligned}$$

The limit above is an easy consequence of the lemma preceding Theorem 7.1.2 in Chung (1968). As the limit function is the moment generating function corresponding to a mean zero normal distribution having variance  $1/(2\rho-1)$ , this must be the asymptotic distribution for  $\sqrt{n}(S_n - \sum_{k=1}^n (\frac{k}{n})^{\rho-1} \frac{1}{n})$ . But

$$Z_n = \sqrt{n}(S_n - 1/\rho) = \sqrt{n}(S_n - \sum_{k=1}^n (\frac{k}{n})^{\rho-1} \frac{1}{n}) + R_n$$

where

$$\begin{aligned}
R_n &= \sqrt{n} \left( \sum_{k=1}^n \left( \frac{k}{n} \right)^{\rho-1} \frac{1}{n} - 1/\rho \right). \\
&= \sqrt{n} \left( \sum_{k=1}^n \left( \frac{k}{n} \right)^{\rho-1} \frac{1}{n} - \int_0^1 x^{\rho-1} dx \right).
\end{aligned}$$

Because  $\rho \geq 1$  we have  $\left(\frac{k}{n}\right)^{\rho-1} \geq x^{\rho-1} \geq \left(\frac{k-1}{n}\right)^{\rho-1}$  whenever  $\frac{k}{n} \geq x \geq \frac{k-1}{n}$ ,  
whence

$$\begin{aligned}
0 \leq R_n &\leq \sqrt{n} \sum_{k=1}^n \left\{ \left( \frac{k}{n} \right)^{\rho-1} \frac{1}{n} - \left( \frac{k-1}{n} \right)^{\rho-1} \frac{1}{n} \right\} \\
&= \sqrt{n} \left( \frac{1}{n} \right) \text{ (Collapsing sum.)}
\end{aligned}$$

Thus  $R_n \rightarrow 0$  as  $n \rightarrow \infty$  and it follows by Slutsky's theorem (Cramer, 1945 Theorem 20.6, p. 254) that  $Z_n = \sqrt{n} (S_n - 1/\rho)$  asymptotically shares the same distribution as  $\sqrt{n} (S_n - \sum_{k=1}^n \left(\frac{k}{n}\right)^{\rho-1} \frac{1}{n})$ . That is,  $Z_n$  is asymptotically normally distributed with mean zero and variance  $1/(2\rho-1)$  as claimed.

It is often suggestive to write this result in the form

$$S_n = 1/\rho + Z_n/\sqrt{n}$$

where  $Z_n$  has a limiting zero mean normal distribution with variance equal to  $1/(2\rho-1)$ .

For what follows, let us define  $M(s)$  to be the number of unfailed fibers at time  $s$  assuming the unit nominal load per fiber  $L = 1$  and power law coefficient  $\kappa = 1$ . Then

$$\begin{aligned}
M(s) &= n && \text{for } 0 \leq s < S_1, \\
&= n-1 && \text{for } S_1 \leq s < S_2, \\
&\vdots && \\
&= 1 && \text{for } S_{n-1} \leq s < S_n, \\
&= 0 && \text{for } S_n \leq s,
\end{aligned}$$

and  $Y_k$  is the sojourn time for the process in state  $k$ . The independent exponential distributions for  $Y_n, Y_{n-1}, \dots, Y_1$  show that  $\{M(s); s \geq 0\}$  is a pure death stochastic process with death rates

$$\Pr\{M(t+\Delta t) = k-1 | M(t) = k\} = n^\rho k^{1-\rho} \Delta t + o(\Delta t),$$

and  $S_n = \inf\{s \geq 0; M(s) = 0\}$  is the first time this process hits zero.

Time varying loads. Consider now a bundle of  $n$  fibers following power law breakdown  $K(l) = \kappa l^\rho$  and subjected to a total load  $nL(t)$  that may vary with time. To avoid trivialities, assume that  $L(t)$  is strictly positive for positive  $t$ , define

$$H(w|L) = \kappa \int_0^w L(\tau)^\rho d\tau$$

and let

$$G(s|L) = H^{-1}(s|L)$$

be the inverse to the strictly increasing function  $H(\cdot|L)$ . For typographical convenience we will often omit the load  $L$  from our notation and simply write  $H(w)$  for  $H(w|L)$  and  $G(s)$  for  $G(s|L)$ .

We will relate the time varying load problem to the constant load problem by using  $H$  as a time scale change. Let  $N(t)$  be the number of unfailed fibers at time  $t$ . Introduce the rescaled process

$$M(s) = N(t) \text{ where } s = H(t|L).$$

Evaluate

$$\begin{aligned}
 \Pr\{M(s+ds) = k-1 | M(s) = k\} &= \Pr\{N(t+dt) = k-1 | N(t) = k\} \\
 &= k \kappa (nL(t)/k)^\rho dt \\
 &= n^\rho k^{1-\rho} ds, \text{ since } ds = \kappa L(t)^\rho dt,
 \end{aligned}$$

to discover that  $M(s)$  evolves as a bundle of fibers carrying unit nominal load and for which  $\kappa = 1$ . If  $W_n$  denotes the bundle failure time under the time varying load, then  $S_n = H(W_n | L)$  is the failure time of a bundle subjected to a constant unit load and we have the explicit representation

$$W_n = G(S_n | L)$$

which relates the bundle failure time  $W_n$  under varying load and arbitrary  $\kappa$  to the failure time  $S_n$  under unit load and with  $\kappa = 1$ .

Under a constant nominal load  $L(t) \equiv L$  we have  $H(w) = \kappa L^\rho w$ , and  $G(s) = s/(\kappa L^\rho)$  so that  $W_n = S_n/(\kappa L^\rho)$  and

$$W_n = 1/(\rho \kappa L^\rho) + Z'_n/\sqrt{n}, \quad (2.4)$$

where  $Z'_n = Z_n/(\kappa L^\rho)$  is asymptotically normally distributed with mean zero and variance equal to  $1/[\kappa L^\rho \sqrt{(2\rho-1)}]^2$ .

This analysis of bundle failure time under deterministic loads and the assumed exponentially distributed failure times of single fibers under constant load is due to Coleman (1958), and later, to Birnbaum and Saunders (1958).



While explicit results concerning bundles of finite size  $n$  are not generally obtainable, the asymptotic normal distribution of bundle failure times prevails under more general circumstances than so far indicated. In particular, suppose that a single fiber subjected to a load  $\ell(s)$  for  $s \geq 0$  follows the failure time distribution

$$F(t|\ell) = \Psi\left(\int_0^t \kappa \ell(s)^\rho ds\right)$$

for some cumulative distribution function  $\Psi$ . Introduce the notation  $g(u) = \Psi^{-1}(u)$  for the inverse function to  $\Psi$ , and  $\phi(y) = (1-y)^\rho$ . Next, let  $S_n$  be the failure time of a bundle of such fibers subjected to a unit nominal load per fiber and for which  $\kappa = 1$ . For a broad class of functions  $\Psi$ , Phoenix (1977) has shown that, asymptotically for large  $n$ , the standardized bundle failure time  $Z_n = \sqrt{n}(S_n - \mu)$  is normally distributed with mean zero and variance  $\sigma^2$ , where

$$\mu = \int_0^1 \phi(y)g'(y)dy$$

$$\sigma^2 = \int_0^1 \int_0^1 \phi'(u)\phi'(v)g'(u)g'(v)(u \wedge v - uv)dudv,$$

with "prime" denoting differentiation.

The most interesting special case concerns the particular function  $\Psi(x) = 1 - \exp\{-cx^\alpha\}$  where  $c$  and  $\alpha$  are fixed positive parameters. In this case, the failure time  $T$  of a single fiber subjected to a constant load  $\ell(t) \equiv L$  follows the Weibull distribution

$$F(t|L) = 1 - \exp\{-c(\kappa L^\rho t)^\alpha\}, \quad t \geq 0,$$

and then

$$\mu = \frac{1}{\alpha c^{1/\alpha}} \int_0^1 (1-y)^{\rho-1} \left\{ \log\left(\frac{1}{1-y}\right) \right\}^{(1-\alpha)/\alpha} dy,$$

$$\sigma^2 = 2 \left( \frac{\rho}{\alpha c^{1/\alpha} \kappa L^\rho} \right)^2 \int_0^1 (1-v)^{\rho-1} \left\{ \log\left(\frac{1}{1-v}\right) \right\}^{(1-\alpha)/\alpha} \int_0^v u (1-u)^{\rho-2} \left\{ \log\left(\frac{1}{1-u}\right) \right\}^{(1-\alpha)/\alpha} du dv.$$

The Weibull family possesses several properties that are desirable for distributions of time to failure or strength of filaments. In particular, if the time to failure of a typical segment of a filament follows a Weibull distribution, then the time to failure of a filament comprised of several independent segments will also follow a (different) Weibull distribution. For a review of the Weibull distribution in this context, see Harlow, Smith and Taylor (1978), which contains other material as well.

When  $c = \alpha = 1$  we return to the exponential case treated in such detail in the beginning of this section, and then  $\mu = 1/\rho$  and  $\sigma^2 = 1/(2\rho-1)$ , as may be verified easily.

Returning to the general Weibull distribution  $\Psi(x) = 1 - \exp\{-cx^\alpha\}$ , if  $W_n$  is the failure time of a bundle having the time varying nominal load per fiber of  $L(s)$ ,  $s \geq 0$ , and whose breakdown rule has an arbitrary value  $\kappa$ , then again we have the representation

$$W_n = G(S_n | L)$$

The crucial assumption here is the postulated power law breakdown rule  $K(l) = \kappa l^\rho$ .



### 3. A simple model with random loads.

Now suppose that the nominal load per fiber  $L(t)$  is a nonnegative stationary process with mean  $E[L(t)] = L$ . We concentrate in this section on the model where-in  $L(t)$  has only the two possible values  $L+\Delta$  and  $L-\Delta$ , with  $0 < \Delta < L$ , and where the sojourn time in each state is exponentially distributed with parameter  $\lambda$ . This process is stationary provided that we stipulate the initial probabilities  $\Pr\{L(0) = L \pm \Delta\} = 1/2$ .

Set  $a_+ = \kappa(L+\Delta)^0$  and  $a_- = \kappa(L-\Delta)^0$ . Figure 1 shows a typical path of  $H(w) = H(w|L)$ , given that  $L(0) = L + \Delta$ .

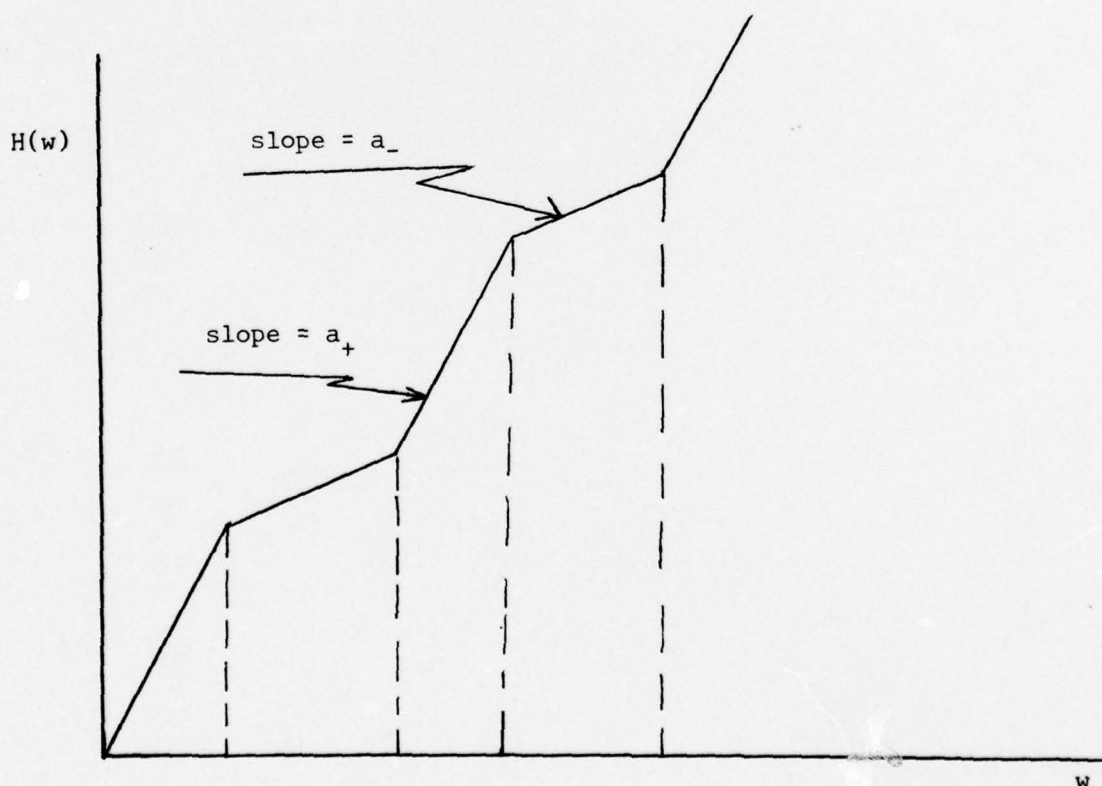


FIGURE 1

Given  $L(0) = L + \Delta$ , the process  $H(w)$  increases linearly at rate  $a_+$  for a duration which is exponentially distributed with parameter  $\lambda$ . It then increases linearly at rate  $a_-$  for a similarly distributed time, and so on. The inverse function  $G(s) = H^{-1}(s)$  has a similar, but distinct, behavior. It increases at rate  $1/a_+$  for an exponentially distributed time having parameter  $\lambda/a_+$ . It then increases at rate  $1/a_-$  for an exponentially distributed time having parameter  $\lambda/a_-$ , and so on. We have the representation

$$G(s) = \int_0^s Y(\sigma) d\sigma$$

where  $\{Y(\sigma); \sigma \geq 0\}$  in a two-state Markov process, the states being  $1/a_+ = 1/[\kappa(L+\Delta)^0]$  and  $1/a_- = 1/[\kappa(L-\Delta)^0]$  and the times in these states being exponentially distributed with respective parameters  $\lambda/a_+$  and  $\lambda/a_-$ . The transition function for  $Y$  is known to be

$$P_{--}(\sigma) = \frac{a_-}{a_+ + a_-} + \frac{a_+}{a_+ + a_-} \exp\{-\lambda(\frac{1}{a_+} + \frac{1}{a_-})\sigma\},$$

and

$$P_{+-}(\sigma) = \frac{a_-}{a_+ + a_-} - \frac{a_-}{a_+ + a_-} \exp\{-\lambda(\frac{1}{a_+} + \frac{1}{a_-})\sigma\}.$$

(See Karlin and Taylor (1975) Problem 7, page 154.) Since  $Y(0) = 1/a_+$  or  $1/a_-$  with probability  $1/2$  each, we have

$$\begin{aligned}
P_-(\sigma) &= \Pr\{Y(\sigma) = 1/a_-\} \\
&= \frac{1}{2}(P_{+-}(\sigma) + P_{--}(\sigma)) \\
&= \frac{a_-}{a_+ + a_-} + \frac{1}{2} \left( \frac{a_+ - a_-}{a_+ + a_-} \right) \exp\left\{-\lambda \left( \frac{1}{a_+} + \frac{1}{a_-} \right) \sigma\right\}
\end{aligned}$$

and

$$\begin{aligned}
P_+(\sigma) &= 1 - P_-(\sigma) \\
&= \frac{a_+}{a_+ + a_-} - \frac{1}{2} \left( \frac{a_+ - a_-}{a_+ + a_-} \right) \exp\left\{-\lambda \left( \frac{1}{a_+} + \frac{1}{a_-} \right) \sigma\right\}.
\end{aligned}$$

Then

$$\begin{aligned}
E[Y(\sigma)] &= \frac{1}{a_+} P_+(\sigma) + \frac{1}{a_-} P_-(\sigma) \\
&= \frac{2}{a_+ + a_-} + \frac{1}{2} \left( \frac{1}{a_-} - \frac{1}{a_+} \right) \left( \frac{a_+ - a_-}{a_+ + a_-} \right) \exp\left\{-\lambda \left( \frac{1}{a_+} + \frac{1}{a_-} \right) \sigma\right\},
\end{aligned}$$

which we abbreviate in the form  $E[Y(\sigma)] = A + Be^{-C\sigma}$  with

$$A = 2/(a_+ + a_-), \quad (3.1)$$

$$B = \frac{1}{2} \left( \frac{1}{a_-} - \frac{1}{a_+} \right) \left( \frac{a_+ - a_-}{a_+ + a_-} \right), \quad (3.2)$$

and

$$C = \lambda \left( \frac{1}{a_+} + \frac{1}{a_-} \right). \quad (3.3)$$

Then, since

$$W_n = \int_0^S n Y(\sigma) d\sigma \quad (3.4)$$

we have

$$\begin{aligned}
 E[W_n] &= E\{E[W_n | S_n]\} \\
 &= E\left\{\int_0^{S_n} (A + B e^{-C\sigma}) d\sigma\right\} \\
 &= A E[S_n] + (B/C)\{1 - E[e^{-CS_n}]\}. \quad (3.5)
 \end{aligned}$$

In conjunction with equations (2.1) and (2.3), this provides an explicit formula for the mean time to bundle failure under the random load  $L$ .

While Equation (3.5) is explicit, it never-the-less remains clumsy. The formula simplifies in certain extreme cases, however, which enables us to gain some insight as to how random loads affect mean bundle failure times. When  $\lambda$  becomes small, the frequency of the load fluctuations decreases and in the limit, when  $\lambda = 0$ , the load process remains at whichever level it began. We then have

$$\begin{aligned}
 \lim_{\lambda \rightarrow 0} E[W_n] &= (A + B)E[S_n] \\
 &= \left\{ \frac{2}{a_+ + a_-} + \frac{1}{2} \left( \frac{1}{a_-} - \frac{1}{a_+} \right) \left( \frac{a_+ - a_-}{a_+ + a_-} \right) \right\} E[S_n].
 \end{aligned}$$

If we postulate that  $\Delta$  is small compared to the base load  $L$ , and that  $\rho$  is moderate so that  $0 < \rho\Delta \ll L$ , we have the Taylor series expansions

$$a_{\pm} = \kappa(L \pm \Delta)^{\rho} \approx \kappa L^{\rho} \left( 1 \pm \rho\Delta/L + \frac{\rho(\rho-1)}{2} \left( \frac{\Delta}{L} \right)^2 \right)$$

and



$$\begin{aligned} \lim_{\lambda \rightarrow 0} E[W_n] &\approx \left(\frac{1}{\kappa L^\rho}\right) \left\{ \frac{1}{1 + \frac{\rho(\rho-1)}{2} \left(\frac{\Delta}{L}\right)^2} + \left(\frac{\rho\Delta}{L}\right)^2 \right\} E[S_n] \\ &\approx (1/\kappa L^\rho) \left\{ 1 + \frac{1}{2} \rho(\rho+1) (\Delta/L)^2 \right\} E[S_n], \quad 0 < \rho\Delta < L. \quad (3.6) \end{aligned}$$

Now it is doubtful that the mean failure time is a relevant criterion of reliability in the extreme case at hand since half the time the bundle will encounter the heavy load  $L + \Delta$  and will fail relatively quickly. Nevertheless, recalling that the mean failure time under no random load ( $\Delta = 0$ ) is  $(1/\kappa L^\rho)E[S_n]$ , we see from (3.6) that the randomness has actually increased the mean failure time. That under certain circumstances randomness could actually increase mean failure times was a surprising discovery.

Much more relevant in practice, it would seem, is the case where  $\lambda$  is large since this corresponds to many random fluctuations in the load prior to bundle failure. Letting  $\lambda \rightarrow \infty$  we have

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} E[W_n] &= A E[S_n] \\ &= \frac{2}{a_+ + a_-} E[S_n]. \end{aligned}$$

When  $\rho \geq 1$  we have the convexity inequality that  $(a_+ + a_-)/2 = (\kappa/2)(L + \Delta)^\rho + (\kappa/2)(L - \Delta)^\rho \geq \kappa L^\rho$  whence

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} E[W_n] &= \frac{2}{a_+ + a_-} E[S_n] \\ &\leq (1/\kappa L^\rho) E[S_n]. \end{aligned}$$

The latter being the strength under no random stress, we conclude that rapidly varying random loads always decrease mean failure time.

If we introduce the notion of an equivalent load  $L_{eq}$  where  $L_{eq}^\rho = \frac{1}{2} (L + \Delta)^\rho + \frac{1}{2} (L - \Delta)^\rho$  we observe that a bundle under the random load  $L \pm \Delta$  has a mean failure time equal to that of a bundle under the fixed load  $L_{eq}$  when  $\lambda \rightarrow \infty$ .

Let us highlight from (3.5) that the design formula  $\lim_{\lambda \rightarrow \infty} E[W_n]$   $= A E[S_n] = 1/(\kappa \rho L_{eq}^\rho)$  is conservative in the sense that  $E[W_n] \geq A E[S_n]$  for all  $\lambda$ . As already pointed out, the presence of rapidly fluctuating random loads tends to decrease mean failure time, and it is of major interest to estimate the additional number of fibers that would be needed to restore the loss. An increase to  $n_1$  fibers from  $n_0$  fibers will decrease the equivalent load to  $(n_0/n_1)L_{eq}$  and this is equated to the deterministic design load  $L$  to obtain

$$\frac{n_1}{n_0} = \frac{L_{eq}}{L}.$$

This ratio  $L_{eq}/L$  as a function of  $\Delta/L$  for various values of  $\rho$  is evaluated in the next section, along with the same ratio under some different distributions for the stationary load process.



#### 4. The asymptotic distribution under random loads.

It is possible, on the one hand, to extend the preceding analysis to cover quite general stationary load processes, and on the other, to refine the analysis and obtain some information concerning the random fluctuations in failure time about the asymptotic mean. The key to this further analysis is a far reaching generalization of the central limit theorem due to Stratonovich (1968) and Has'minskii (1966) which we enunciate only in a very limited case that suffices to meet our needs. Let  $\{L(\tau); \tau \geq 0\}$  be a stationary process taking values in the strictly positive bounded interval  $[L_{\min}, L_{\max}]$ . Let

$$L = E[L(\tau)] \quad \text{and} \quad L_{\text{eq}} = \{E[L(\tau)^\rho]\}^{1/\rho}.$$

Let  $F_s^t$  be the  $\sigma$ -algebra of events generated by  $L(\tau)$  for  $s \leq \tau \leq t$  and assume an asymptotic independence for  $L(\tau)$  in the form

$$\sup\{|P(B|A) - P(B)|; A \in F_0^t, B \in F_{t+s}^\infty\} < \beta(s)$$

where  $\beta(s) \rightarrow 0$  as  $s \rightarrow \infty$  sufficiently fast so that  $s^6 \beta(s) \rightarrow 0$  as well.

Let  $F(l)$  be a smooth function of  $l$ , bounded on the interval  $[L_{\min}, L_{\max}]$  and for which  $E[F(L(\tau))] = 0$ , and

$$\lim_{T \rightarrow \infty} \frac{2}{T} \int_0^T \int_0^s E[F(L(s))F(L(\sigma))] d\sigma ds = \delta^2 > 0.$$

Next, for a fixed  $\lambda > 0$ , consider the solution  $Z_\lambda(t)$  to the differential equation

$$dZ_{\lambda}(t)/dt = \sqrt{\lambda} F(L(\lambda t)), Z_{\lambda}(0) = 0.$$

Of course, the solution is  $Z_{\lambda}(t) = \sqrt{\lambda} \int_0^t F(L(\lambda s)) ds = (1/\sqrt{\lambda}) \int_0^{\lambda t} F(L(v)) dv$ . Note that large values for  $\lambda$  correspond to load processes  $L_{\lambda}(s) = L(\lambda s)$  having "many fluctuations" prior to time  $s$ , or alternatively, observing the fixed load process  $L(v)$  over the long duration  $\lambda t$ . That is, large values for  $\lambda$  are quite relevant to many cable design situations. The result of Stratonovich and Has'minskii, restricted to the case at hand, asserts that, as  $\lambda \rightarrow \infty$ , the processes  $\{Z_{\lambda}(t), t \geq 0\}$  converge weakly in the space of continuous functions to a Brownian motion process having variance parameter  $\delta^2$ .

We apply this result using for  $F$  the smooth function

$$F(\ell) = \kappa(\ell^0 - L_{eq}^0), L_{min} \leq \ell \leq L_{max},$$

whence

$$\begin{aligned} Z_{\lambda}(t) &= \sqrt{\lambda} \kappa \left\{ \int_0^t L_{\lambda}(s)^0 ds - L_{eq}^0 t \right\} \\ &= \sqrt{\lambda} \{ H_{\lambda}(t) - \kappa L_{eq}^0 t \} \end{aligned}$$

where  $L_{\lambda}(s) = L(\lambda s)$  and  $H_{\lambda}(t) = \kappa \int_0^t L_{\lambda}(s)^0 ds$ . We conclude that  $\{Z_{\lambda}(t), t \geq 0\}$  is, asymptotically for large  $\lambda$ , a Brownian motion. The next step is to relate this convergence to the behavior of  $G_{\lambda}(s) = H_{\lambda}^{-1}(s)$ . To this end, define

$$V_{\lambda}(s) = \sqrt{\lambda} \{ G_{\lambda}(s) - s/(\kappa L_{eq}^0) \}.$$

A careful study of Figure 2 will reveal that

$$V_{\lambda}(s) = -(1/\kappa L_{eq}^{\rho}) Z_{\lambda}[G_{\lambda}(s)].$$

Now when  $\lambda$  is large, we know via the ergodic theorem for stationary processes that  $H_{\lambda}(t) = \kappa \int_0^t L_{\lambda}(s)^{\rho} ds = \kappa(\frac{1}{\lambda}) \int_0^{\lambda t} L(v)^{\rho} dv$  is near its mean value  $\kappa L_{eq}^{\rho} t$ , and hence, for the inverse, that  $G_{\lambda}(s)$

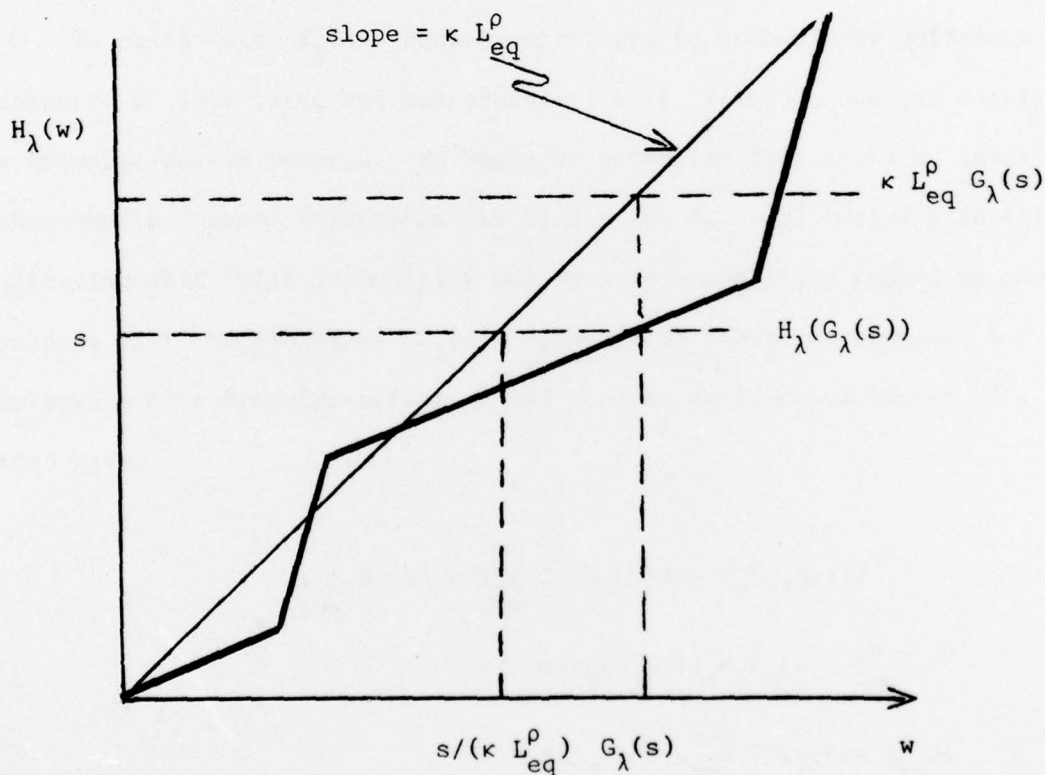


FIGURE 2: Showing that

$$\text{slope} = \kappa L_{eq}^{\rho} = \frac{\kappa L_{eq}^{\rho} G_{\lambda}(s) - H_{\lambda}[G_{\lambda}(s)]}{G_{\lambda}(s) - s/(\kappa L_{eq}^{\rho})}$$

$$\text{or } \sqrt{\lambda}\{G_{\lambda}(s) - s/(\kappa L_{eq}^{\rho})\} = -(1/\kappa L_{eq}^{\rho})\sqrt{\lambda}\{H_{\lambda}[G_{\lambda}(s)] - \kappa L_{eq}^{\rho} G_{\lambda}(s)\}.$$

is near  $s/(\kappa L_{eq}^\rho)$ . To be precise, these convergences take place with probability one uniformly on compact  $s$  intervals. We exploit this by writing

$$V_\lambda(s) = -(1/\kappa L_{eq}^\rho) Z_\lambda(s/\kappa L_{eq}^\rho) - (1/\kappa L_{eq}^\rho) R_\lambda(s)$$

where

$$R_\lambda(s) = Z_\lambda(G_\lambda(s)) - Z_\lambda(s/\kappa L_{eq}^\rho).$$

We claim that  $R_\lambda(s)$  converges to zero in probability uniformly on compact  $s$  intervals, and consequently, that  $V_\lambda(s)$  converges weakly to a Brownian motion process. We begin to establish this claim by invoking Skorodhod's theorem (Skorodhod (1965) Section 6, p. 9) implying in this situation that (with probability one in some probability space) we may suppose that the processes  $Z_\lambda(w)$  converge uniformly on compact  $w$  intervals to a Brownian motion  $Z_\infty(w)$  having variance parameter  $\delta^2$ . We then have

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} R_\lambda(s) &= \lim_{\lambda \rightarrow \infty} \{Z_\lambda(G_\lambda(s)) - Z_\infty(G_\lambda(s))\} \\ &\quad + \lim_{\lambda \rightarrow \infty} \{Z_\infty(G_\lambda(s)) - Z_\infty(s/\kappa L_{eq}^\rho)\} \\ &\quad + \lim_{\lambda \rightarrow \infty} \{Z_\infty(s/\kappa L_{eq}^\rho) - Z_\lambda(s/\kappa L_{eq}^\rho)\} \\ &= 0, \end{aligned}$$

where again the convergence is uniform on compact  $s$  intervals in view of



the inequality  $s/(\kappa L_{\max}^\rho) \leq G_\lambda(s) \leq s/(\kappa L_{\min}^\rho)$ . It follows that

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} V_\lambda(s) &= -(1/\kappa L_{\text{eq}}^\rho) Z_\infty(s/\kappa L_{\text{eq}}^\rho) \\ &= V_\infty(s) \end{aligned}$$

where  $V_\infty(s)$  is a Brownian motion process having variance parameters

$$E[V_\infty(1)^2] = \delta^2/(\kappa L_{\text{eq}}^\rho)^3.$$

As an illustration, we evaluate  $\delta^2$  for the two state Markov load process  $L(\tau)$  that introduced our study of random loads. Recall that the possible states are  $L \pm \Delta$ , and denote the transition function for  $L(\tau)$  by  $Q$  in the form  $Q_{+-}(\tau) = \Pr\{L(\tau) = L - \Delta | L(0) = L + \Delta\}$ , etc. We have

$$Q_{--}(\sigma) = Q_{++}(\sigma) = \frac{1}{2}(1 + e^{-2\lambda\sigma})$$

and

$$Q_{+-}(\sigma) = Q_{-+}(\sigma) = \frac{1}{2}(1 - e^{-2\lambda\sigma}).$$

Since  $L(\tau)^\rho$  is equally likely to begin in either of its states, its mean value is  $E[L(\tau)^\rho] = \frac{1}{2}(L+\Delta)^\rho + \frac{1}{2}(L-\Delta)^\rho = L_{\text{eq}}^\rho$ , and for  $\sigma < \tau$ ,

$$\begin{aligned} E[L(\sigma)^\rho L(\tau)^\rho] &= (1/2\kappa^2)[a_{++}Q_{++}(\tau-\sigma) + a_{+-}Q_{+-}(\tau-\sigma) \\ &\quad + a_{-+}Q_{-+}(\tau-\sigma) + a_{--}Q_{--}(\tau-\sigma)] \end{aligned}$$

which, after subtracting the product of means  $E[L(\tau)^\rho L(\sigma)^\rho] = L_{eq}^{2\rho}$  and simplyfying leads to the covariance

$$\begin{aligned} E[F(L(\sigma))F(L(\tau))] &= \kappa^2 \text{Cov}[L(\sigma)^\rho, L(\tau)^\rho] \\ &= \beta^2 e^{-2\lambda|\tau-\sigma|} \end{aligned}$$

where  $\beta = (a_+ - a_-)/2 = [(L+\Delta)^\rho - (L-\Delta)^\rho]/2$ .

Then

$$\begin{aligned} \int_0^T \int_0^S E[F(L(s))F(L(\sigma))] d\sigma ds &= \beta^2 \int_0^T \int_0^S e^{-2\lambda(s-\sigma)} d\sigma ds \\ &= (\beta^2/2\lambda) \int_0^T (1 - e^{-2\lambda s}) ds \\ &= (\beta^2/2\lambda) \{T + (2\lambda)^{-1}(e^{-2\lambda T} - 1)\} \end{aligned}$$

and

$$\delta^2 = \lim_{T \rightarrow \infty} \frac{2}{T} \int_0^T \int_0^S E[F(L(s))F(L(\sigma))] d\sigma ds = \beta^2/\lambda.$$

That is,  $\delta^2 = \lambda^{-1}[(a_+ - a_-)/2]^2 = \lambda^{-1}[\kappa((L+\Delta)^\rho - (L-\Delta)^\rho)/2]$ .

Returning to the general stationary load process, let us consider large bundles ( $n \rightarrow \infty$ ) and set  $\lambda = \theta n$  so that  $\lambda$  becomes infinite with  $n$ . We have, on the one hand,

$$\sqrt{n} \{S_n - \mu\} = Z_n$$

where  $Z_n$  has a limit normal distribution with mean zero and variance  $\sigma^2$ ; while on the other hand



$$V_\lambda(s) = \sqrt{\theta n} \{G_{\theta n}(s) - s/(\kappa L_{eq}^\rho)\}$$

is converging weakly to a Brownian motion process  $V_\infty$  having variance coefficient  $\delta^2/(\kappa L_{eq}^\rho)^3$ . Since the bundle failure time  $W_n$  is given by  $W_n = G_{\theta n}(S_n)$ , we write

$$\begin{aligned} \sqrt{n}\{W_n - \mu/(\kappa L_{eq}^\rho)\} &= \sqrt{n}\{G_{\theta n}(S_n) - S_n/(\kappa L_{eq}^\rho)\} + \sqrt{n}\{S_n - \mu\}/(\kappa L_{eq}^\rho) \\ &= (1/\sqrt{\theta}) V_{n\theta}(S_n) + Z_n/(\kappa L_{eq}^\rho) \\ &= (1/\sqrt{\theta}) V_{n\theta}(\mu) + Z_n/(\kappa L_{eq}^\rho) + \zeta_n \end{aligned}$$

where

$$\zeta_n = (1/\sqrt{\theta})\{V_{n\theta}(S_n) - V_{n\theta}(\mu)\}.$$

Now  $\lim_{n \rightarrow \infty} S_n = \mu$ , while  $V_{n\theta}$  is converging (uniformly on compact intervals) to the Brownian motion  $V_\infty$ , and  $Z_n$  is converging to the normally distributed random variable  $Z$ . Thus  $\zeta_n \rightarrow 0$  in probability and we have the weak convergence

$$\sqrt{n}\{W_n - \mu/(\kappa L_{eq}^\rho)\} \Rightarrow (1/\sqrt{\theta}) V_\infty(\mu) + Z/(\kappa L_{eq}^\rho).$$

It follows that, asymptotically for large  $n$ , the standardized failure time  $\sqrt{n}\{W_n - \mu/(\kappa L_{eq}^\rho)\}$  is normally distributed with mean zero and variance

$$\frac{\delta^2}{(\kappa L_{eq}^\rho)^3 \theta} + \frac{\sigma^2}{(\kappa L_{eq}^\rho)^2}.$$

We see that random loads add a term to the variance of the failure time in addition to decreasing the asymptotic mean failure time from  $\mu/(\kappa L^\rho)$  to  $\mu/(\kappa L_{eq}^\rho)$ .

Let us suppose that  $n_0$  is the number of fibers needed to achieve a prescribed target mean cable lifetime under a fixed load  $L$ , and that  $n_1$  is the greater number of fibers needed to achieve the same target mean cable life under a random loading  $L(\tau)$  having an associated  $L_{eq} = \{E[L(\tau)^\rho]\}^{1/\rho}$ . As shown earlier, we have the equivalence  $n_1/n_0 = L_{eq}/L$  so that the ratio  $L_{eq}/L$  provides some measure of the increased "cost" associated with random loads. This ratio is sensitive to the fiber breakdown rule parameter  $\rho$  and the marginal distribution of the load process as we shall see.

Let  $\Delta^2 = E[(L(\tau) - L)^2]$  be the variance of the load process. When  $\Delta$  is very small relative to  $L$  and  $\rho$  is moderate, so that  $\rho\Delta$  remains small, we have the Taylor series approximation

$$\begin{aligned} L_{eq}/L &= \{E[(L(\tau)/L)^\rho]\}^{1/\rho} \\ &= \{E[(1 + \frac{L(\tau)-L}{L})^\rho]\}^{1/\rho} \\ &\approx \{1 + \frac{1}{2}\rho(\rho-1)(\frac{\Delta}{L})^2\}^{1/\rho}. \end{aligned}$$

On the other hand, suppose that  $L_{max}$  is the essential supremum of  $L(\tau)$  in that  $\Pr\{L(\tau) > L_{max} - \epsilon\} > 0$  for any  $\epsilon > 0$  while  $\Pr\{L(\tau) \leq L_{max}\} = 1$ . Then, as is well known (see e.g. Taylor, A.E. (1958) p. 91)

$$\lim_{\rho \rightarrow \infty} L_{eq} = \lim_{\rho \rightarrow \infty} \{E[L(\tau)^\rho]\}^{1/\rho} = L_{max}.$$

That is, the behavior of  $L_{eq}$  for large values of  $\rho$  is quite distinct from that at small values. To obtain a better understanding of the effect of  $\rho$  on the ratio  $L_{eq}/L$ , explicit formulas have been derived in several special cases.

(I) Recall that when  $L(\tau) = L \pm \Delta$ , each with probability  $1/2$ , then

$$L_{eq}/L = \left\{ \frac{1}{2}(1+\Delta/L)^\rho + \frac{1}{2}(1-\Delta/L)^\rho \right\}^{1/\rho}.$$

(II) Suppose that  $L(\tau)$  is normally distributed with mean  $L$  and variance  $\Delta^2$ . Then for  $\rho = 2m$ , where  $m = 1, 2, \dots$  we have

$$\begin{aligned} L_{eq}/L &= \left\{ E\left[ \left( 1 + \frac{L(\tau)-L}{L} \right)^\rho \right] \right\}^{1/\rho} \\ &= \left\{ \sum_{k=0}^{\rho/2} \binom{\rho}{2k} E\left[ \left( \frac{L(\tau)-L}{L} \right)^{2k} \right] \right\}^{1/\rho} \\ &= \left\{ 1 + \sum_{k=1}^{\rho/2} \frac{\rho!}{2(k!)(\rho-2k)!} (\Delta/L)^{2k} \right\}^{1/\rho}. \end{aligned}$$

(III) Suppose  $L(\tau)$  is uniformly distributed on the interval  $[L-A, L+A]$ . Then  $E[L(\tau)] = L$  and  $\text{Var}[L(\tau)] = \Delta^2 = A^2/3$ , while

$$E[L(\tau)^\rho] = \frac{1}{2A} \int_{L-A}^{L+A} x^\rho dx = \frac{(L+A)^{1+\rho} - (L-A)^{1+\rho}}{2A(1+\rho)}.$$

Then

$$\begin{aligned} L_{eq}/L &= \left\{ \frac{L}{2A(1+\rho)} [(1+A/L)^{1+\rho} - (1-A/L)^{1+\rho}] \right\}^{1/\rho} \\ &= \left\{ - \frac{L}{2\sqrt{3}\Delta(1+\rho)} [(1 + \sqrt{3}\Delta/L)^{1+\rho} - (1 - \sqrt{3}\Delta/L)^{1+\rho}] \right\}^{1/\rho}. \end{aligned}$$

$L_{eq}/L$  as a function of the coefficient of variation  $\Delta/L$  for various values of  $\rho$  and distributions I, II and III is depicted in Figure 3. It is readily apparent that the ratio depends strongly on the form of the distribution of the random loads and cannot, for example, be adequately summarized by the mean and variance of this distribution alone, even at modest values of  $\Delta/L$ . This dependence on the load distribution becomes more acute at the higher values of  $\rho$ .

It is clear that parallel systems whose elements exhibit fatigue of the power law breakdown type should be designed with caution when randomly varying loads may be encountered.



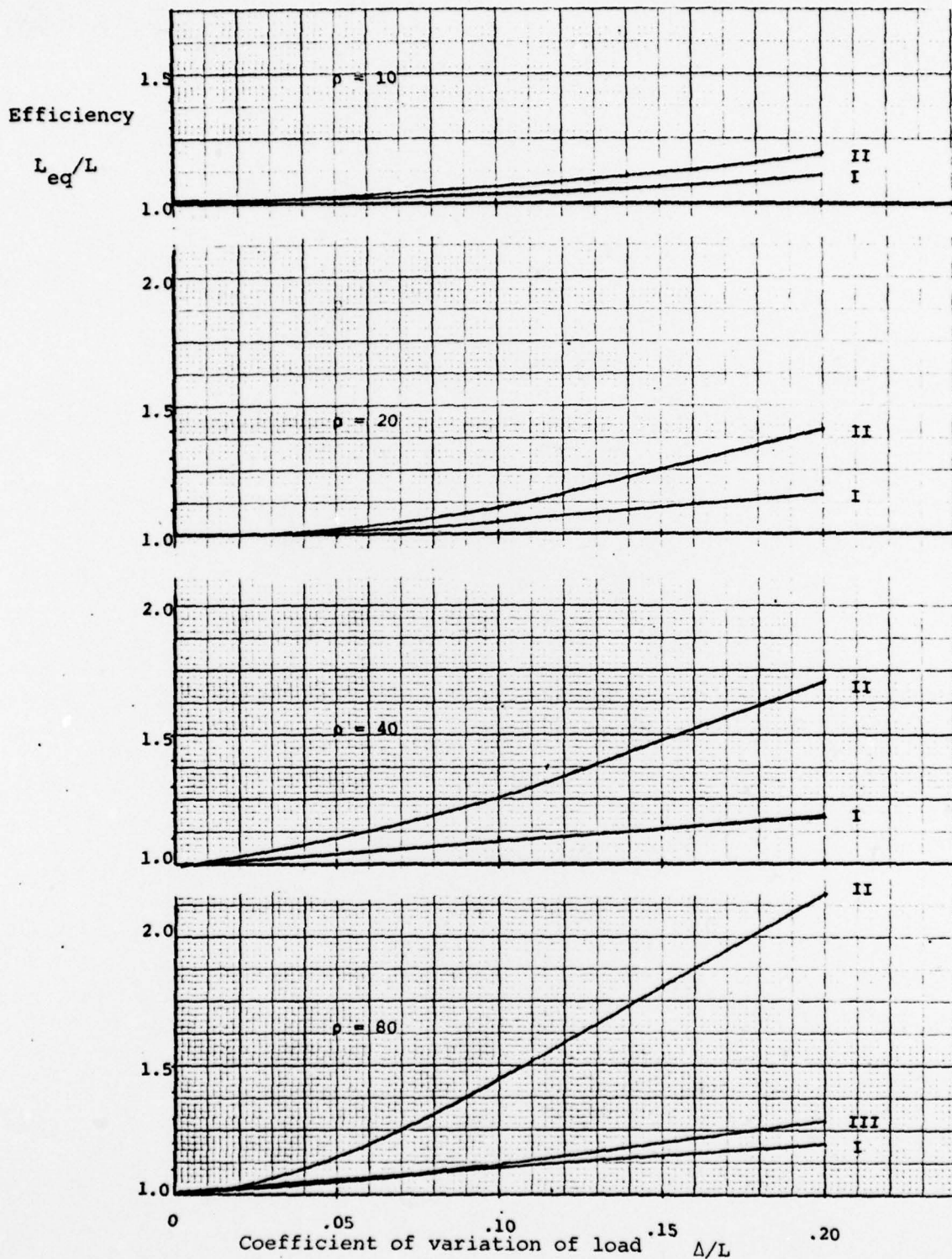


Figure 3: The loss in efficiency due to random loads as measured by  $L_{eq}/L$  versus the coefficient of variation  $\Delta/L$  of the random load process for various values of the fatigue parameter  $\rho$ . Curves I, II and III correspond to two-valued loads, normally distributed loads and uniformly distributed loads, respectively.

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concerning the distribution of bundle or cable failure time, and especially the mean failure time, are obtained. Where the fluctuations in load are frequent relative to cable life, such as may occur in long-lived cables, it is shown that randomness in load tends to decrease mean cable life, but it is suggested that the reduction in mean life often can be restored by modestly reducing the base load on the structure or by modestly increasing the number of elements in the cable.

In later pages this simple model is extended to cover a broader range of materials and random loadings. Asymptotic distributions and mean failure times are given for cable elements that follow a Weibull distribution of failure time under constant load, and loads that are general nonnegative stationary processes subject only to some mild condition of asymptotic independence. When the power law breakdown exponent is large, the mean time to cable failure depends heavily on the exact form of the marginal probability distribution for the random load process and cannot be summarized by the first two moments of this distribution alone.